

On the classical κ -particle

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Abstract

The dependence of velocity on momentum for the free massive particle obeying the κ -Poincaré Poisson symmetry is calculated in terms of intrinsic non-commuting space-time coordinates and shown to have a monotonic character, with upper limit of velocity equal to 1.

Introduction

The theory of quantum groups [1] (and Poisson groups [2, 3, 4, 5]) provides us with ‘non-commutative’ models of space-time which (depending on some deformation parameter) can be arbitrarily close to the usual ‘commutative’ model. In this situation it is rather reasonable to study how the known models of physical systems behave when going to the non-commutative case.

An idea how to construct a model of a classical (free) particle ‘moving’ in a Poisson Minkowski space-time M has been given in [6] (see also [7]). The natural geometric way is to replace the cotangent bundle to M (the *extended phase space* with coordinates q^k, p_l , $k, l = 0, 1, 2, 3$) by the symplectic groupoid of the Poisson manifold M (called also the *twisted cotangent bundle* of M , or the *phase space* of M , see [6, 7, 9, 10] and references therein), which we denote by $\mathbf{Ph} M$. Then the Poisson action of the underlying Poisson Poincaré group G on M yields so called *moment map* from $\mathbf{Ph} M$ to the dual Poisson group G^* of G . Taking the inverse image of a symplectic leaf (an orbit of the dressing action) in G^* by the moment map, we obtain the *mass shell* in this new context.

The characteristics on the mass shell (given by the directions of degeneracy of the symplectic form on the shell) are interpreted as phase trajectories of the particle. We can project these characteristics down to M , even if the interpretation of these ‘world lines’ is not quite clear. Note that there are two projections in the ‘twisted cotangent bundle’ and one can hesitate which to use. One should probably choose always the left projection (recall that it is the left projection which reproduces the Poisson structure on M : the Poisson bracket of functions on M coincides with their bracket in $\mathbf{Ph} M$ when pulled back by the left projection) and to admit that each model has its ‘mirror image’. Anyway, these ‘world lines’ exist in the model.

In [6] we have calculated phase trajectories and world lines in the $D = 2$ case. In [7] we have considered in the same model another set of position variables (another projection

in $\mathbf{Ph}M$). These variables, in contrast to the previous ones, were Poisson commuting, although introduced somewhat ‘by hands’. The world lines obtained using these non-intrinsic ‘commuting positions’ seemed to be even more ‘realistic’ than the previous ones.

An opposite situation seems to happen in the model based on the Poisson Poincaré group [11] corresponding to the κ -deformation [12, 13] in $D = 4$. In the present paper we calculate the world lines in terms of non-commuting original positions for this model. We show that the dependence of the velocity on the momentum is monotonic (with the upper limit = 1), in contrast with the unpleasant behavior observed recently in [14], when using commuting positions which appear also in this model.

1 The twisted cotangent bundle to κ -Minkowski

Let G denote the Poincaré group. The Poisson structure on G corresponding to the κ -deformation is fixed by the classical r -matrix

$$r = h(L_1 \wedge P_1 + L_2 \wedge P_2 + L_3 \wedge P_3), \quad h \equiv \frac{1}{\kappa}$$

(cf. [11]) on the Poincaré Lie algebra, where L_i, P_j denote the ‘boost’ generators and the spatial translations generators, respectively. The corresponding Poisson Minkowski space is the usual Minkowski space M (which we may identify with the subgroup $V \subset G$ of translations), with the following Poisson structure

$$\{q^k, q^0\} = hq^k, \quad \{q^k, q^l\} = 0, \quad k, l = 1, 2, 3. \quad (1)$$

Due to the linearity of the Poisson bracket, M is the dual space of a Lie algebra \mathfrak{k} :

$$M \cong V \cong \mathfrak{k}^*.$$

It follows that $\mathbf{Ph}M = \mathbf{Ph}\mathfrak{k}^* = T^*K$, where K is the group having \mathfrak{k} as its Lie algebra [8]. The group K has an obvious semi-direct product structure. It is convenient to realize this group as $\mathbb{R}^3 \times \mathbb{R}$ with the group multiplication given by

$$(\vec{p}, p_0)(\vec{p}', p'_0) = \left(\vec{p} \exp\left(\frac{h}{2}p'_0\right) + \exp\left(-\frac{h}{2}p_0\right) \vec{p}', p_0 + p'_0 \right), \quad (2)$$

because then K is easily recognized as the quotient group of G^* described in [11], corresponding to the momentum coordinates P_0, P_1, P_2, P_3 (as it should be, to the Poisson inclusion $\mathfrak{k}^* \subset G$ there corresponds dually the Poisson projection $G^* \rightarrow K$).

According to the general rule, we should now compute the moment map $J: \mathbf{Ph}M \rightarrow G^*$ implied by the Poisson action of G on M . However, in order to calculate the mass shell in $\mathbf{Ph}m$ it is sufficient to compute only the translational momentum part $\mathbf{Ph}M \rightarrow G^* \rightarrow K$ of J , because the first *Casimir* [12, 13] (valid also in the Poisson context),

$$C_1 = \left(\frac{\sinh \frac{h}{2}p_0}{\frac{h}{2}} \right)^2 + \sum_{k=1}^3 p_k p^k, \quad p^k \equiv -p_k, \quad (3)$$

depends only on p 's (this standard form of C_1 holds in the parametrization of K as in (2)). But the projected J is nothing else but the moment map for the action restricted to the subgroup $V \subset G$. Since the action of V on $M = V = \mathfrak{k}^*$ is just the addition, the corresponding moment map is just the projection $T^*K \rightarrow K$ in the cotangent bundle.

We conclude that the mass shell in $\mathbf{Ph} M = T^*K$ for the mass m is given by

$$\{(p, x) \in T^*K : C_1(p) = m^2\}.$$

Here $p \in K \cong \mathbb{R}^3 \times \mathbb{R}$ and x denotes the dual (canonically conjugate) variable (assume it takes values in $\mathbb{R}^3 \times \mathbb{R}$).

2 Trajectories

The equations of characteristics on the mass shell are obtained simply by calculating the flow of C_1 in $\mathbf{Ph} M$:

$$\dot{x}^k = \left\{ \frac{1}{2} C_1, x^k \right\} = p^k, \quad \dot{x}^0 = \left\{ \frac{1}{2} C_1, x^k \right\} = \frac{\sinh hp_0}{h}, \quad k = 1, 2, 3.$$

As shown in [14], the velocity

$$\frac{dx^k}{dx^0} = \frac{p^k}{\frac{\sinh hp_0}{h}} = \frac{p^k}{\sqrt{p^2 + m^2} \sqrt{1 + \left(\frac{h}{2}\right)^2 (p^2 + m^2)}}$$

is not a monotonic function of the momentum. However, the variables x are here not the natural position coordinates. The intrinsic coordinates implied by the structure of the twisted cotangent bundle come from the left projection in $\mathbf{Ph} M$, which in our case is just the right translation of a covector in T^*K to the group unit [8]:

$$q^k = x^k \exp\left(\frac{h}{2} p_0\right), \quad q^0 = x^0 - \frac{h}{2} \sum_{j=1}^3 p_j x^j. \quad (4)$$

One can easily check that these coordinates have Poisson brackets exactly as in (1). The flow of C_1 gives now

$$\dot{q}^k = p^k \exp\left(\frac{h}{2} p_0\right), \quad \dot{q}^0 = \frac{\sinh hp_0}{h} + \frac{h}{2} \vec{p}^2,$$

hence the velocity in terms of these coordinates has the following square

$$\left(\frac{d\vec{q}}{dq^0} \right)^2 = \frac{\left(\exp(hp_0) - 1 \right)^2 - m^2 h^2 \exp(hp_0)}{\left(\exp(hp_0) - 1 - \frac{m^2 h^2}{2} \right)^2}. \quad (5)$$

Using the monotonic change of variables $t \equiv \exp(hp_0) - 1$, we obtain

$$\left(\frac{d\vec{q}}{dq^0} \right)^2 = f(t) = \frac{t^2 - m^2 h^2 (t + 1)}{\left(t - \frac{m^2 h^2}{2} \right)^2}. \quad (6)$$

An elementary calculus shows that

1. $\lim_{t \rightarrow \infty} f(t) = 1$ (from below)
2. $f'(t_0) = 0 \iff t_0 = \frac{m^4 h^4 / 2}{4 - m^2 h^2}$.

From these facts it follows that f is a monotonic function, with the limit at infinity equal 1.

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